COMBINATORIAL AND MODEL-THEORETICAL PRINCIPLES RELATED TO REGULARITY OF ULTRAFILTERS AND COMPACTNESS OF TOPOLOGICAL SPACES. II.

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Abstract. We find many conditions equivalent to the model-theoretical property $\lambda \stackrel{\kappa}{\Rightarrow} \mu$ introduced in [L1]. Our conditions involve uniformity of ultrafilters, compactness properties of products of topological spaces and the existence of certain infinite matrices.

See Part I [L7] or [CN, CK, KM, KV, HNV] for unexplained notation.

According to [L1], if $\lambda \geq \mu$ are infinite regular cardinals, and κ is a cardinal, $\lambda \stackrel{\kappa}{\Rightarrow} \mu$ means that the model $\langle \lambda, <, \gamma \rangle_{\gamma<\lambda}$ has an expansion $\mathfrak{A}$ in a language with at most κ new symbols such that whenever $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B}$ has an element x such that $\mathfrak{B} \models \gamma < x$ for every $\gamma < \lambda$, then $\mathfrak{B}$ has an element y such that $\mathfrak{B} \models \alpha < y < \mu$ for every $\alpha < \mu$.

An ultrafilter D over λ is said to be uniform if and only if every member of D has cardinality λ. If λ is a regular cardinal, then it is obvious that an ultrafilter D is uniform over λ if and only if the interval $[0, \gamma] \notin D$, for every $\gamma < \lambda$, if and only if the interval (γ, λ) is in D, for every $\gamma < \lambda$.

Thus, if D is an ultrafilter over some regular cardinal λ, and if Id_D denotes the D-class of the identity function on λ, then D is uniform over λ if and only if in the model $\mathfrak{C} = \prod_D \mathfrak{A}$ we have that $d(\gamma) < Id_D$ for every $\gamma < \lambda$. Here, d denotes the elementary embedding.

If D is an ultrafilter over I, and $f : I \to J$, then $f(D)$ is the ultrafilter over J defined by: $Y \in f(D)$ if and only if $f^{-1}(Y) \in D$.

If κ, λ are infinite cardinals, a topological space is said to be $[\kappa, \lambda]$-*compact* if and only if every open cover by at most λ sets has a subcover

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by less than κ sets. No separation axiom is needed to prove the results of the present paper.

Theorem 1. *Suppose that $\lambda \geq \mu$ are infinite regular cardinals, and $\kappa \geq \lambda$ is an infinite cardinal. Then the following conditions are equivalent.*

(a) $\lambda \overset{\kappa}{\Rightarrow} \mu$ holds.

(b) There are κ functions $(f_\beta)_{\beta<\kappa}$ from λ to μ such that whenever D is an ultrafilter uniform over λ then there exists some $\beta < \kappa$ such that $f_\beta(D)$ is uniform over μ.

(b′) There are κ functions $(f_\beta)_{\beta<\kappa}$ from λ to μ for which the following holds: for every function $g : \kappa \to \mu$ there exists some finite set $F \subseteq \kappa$ such that $\left|\bigcap_{\beta\in F} f_\beta^{-1}([0, g(\beta)))\right| < \lambda$.

(c) There is a family $(B_{\alpha,\beta})_{\alpha<\mu,\beta<\kappa}$ of subsets of λ such that:

(i) For every $\beta < \kappa$, $\bigcup_{\alpha<\mu} B_{\alpha,\beta} = \lambda$;

(ii) For every $\beta < \kappa$ and $\alpha \leq \alpha' < \mu$, $B_{\alpha,\beta} \subseteq B_{\alpha',\beta}$;

(iii) For every function $g : \kappa \to \mu$ there exists a finite subset $F \subseteq \kappa$ such that $|\bigcap_{\beta\in F} B_{g(\beta),\beta}| < \lambda$.

(d) Whenever $(X_\beta)_{\beta<\kappa}$ is a family of topological spaces such that no X_β is $[\mu, \mu]$-compact, then $X = \prod_{\beta<\kappa} X_\beta$ is not $[\lambda, \lambda]$-compact.

(e) The topological space μ^κ is not $[\lambda, \lambda]$-compact, where μ is endowed with the topology whose open sets are the intervals $[0, \alpha)$ $(\alpha \leq \mu)$, and μ^κ is endowed with the Tychonoff topology.

Remark 2. An analogue of Theorem 1 holds for the more general notion $(\lambda, \mu) \overset{\kappa}{\Rightarrow} (\lambda', \mu')$ introduced in [L2] (see also [L3, Section 0]). Details shall be presented elsewhere. For this more general notion, the equivalence of conditions analogue to (a) and (b) above has been stated in [L5]. There we also stated the analogue of (b) $\Rightarrow$ (d).

Proof. (a) $\Rightarrow$ (b). Let $\mathfrak{A}$ be an expansion of $\langle\lambda, <, \gamma\rangle_{\gamma<\lambda}$ witnessing $\lambda \overset{\kappa}{\Rightarrow} \mu$.

Without loss of generality we can assume that $\mathfrak{A}$ has Skolem functions (see [CK, Section 3.3]). Indeed, since $\kappa \geq \lambda$, adding Skolem functions to $\mathfrak{A}$ involves adding at most κ new symbols.

Consider the set of all functions $f : \lambda \to \mu$ which are definable in $\mathfrak{A}$. Enumerate them as $(f_\beta)_{\beta<\kappa}$. We are going to show that these functions witness (b).

Indeed, let D be an ultrafilter uniform over λ. Consider the D-class Id_D of the identity function on λ. Since D is uniform over λ, in the model $\mathfrak{C} = \prod_D \mathfrak{A}$ we have that $d(\gamma) < Id_D$ for every $\gamma < \lambda$, where d denotes the elementary embedding. Let $\mathfrak{B}$ be the Skolem hull of Id_D

in $\mathfrak{C}$. By Łoš Theorem, $\mathfrak{C} \equiv \mathfrak{A}$. Since $\mathfrak{A}$ has Skolem functions, $\mathfrak{B} \equiv \mathfrak{C}$ [CK, Proposition 3.3.2]. By transitivity, $\mathfrak{B} \equiv \mathfrak{A}$.

Since $\mathfrak{A}$ witnesses $\lambda \overset{\kappa}{\Rightarrow} \mu$, then $\mathfrak{B}$ has an element y_D such that $\mathfrak{B} \models \alpha < y_D < \mu$ for every $\alpha < \mu$.

Since $\mathfrak{B}$ is the Skolem hull of Id_D in $\mathfrak{C}$, we have $y_D = f(Id_D)$, that is, $y_D = f_D$, for some function $f : \lambda \to \lambda$ definable in $\mathfrak{A}$. Since f is definable, then also the following function f' is definable:

$$f'(\gamma) = \begin{cases} f(\gamma) & \text{if } f(\gamma) < \mu \\ 0 & \text{if } f(\gamma) \geq \mu \end{cases}$$

Since $\mathfrak{B} \models y_D < \mu$, then $\{\gamma < \lambda | y(\gamma) < \mu\} \in D$. Since $y_D = f_D$, $\{\gamma < \lambda | y(\gamma) = f(\gamma)\} \in D$. Hence, $\{\gamma < \lambda | y(\gamma) = f'(\gamma)\} \in D$, being larger than the intersection of two sets in D. Thus, $y_D = f'_D$.

Since $f' : \lambda \to \mu$ and f' is definable in $\mathfrak{A}$, then $f = f_\beta$ for some $\beta < \kappa$, thus $y_D = (f_\beta)_D$.

We need to show that $D' = f_\beta(D)$ is uniform over μ. Indeed, for every $\alpha_0 < \mu$, and since $\mathfrak{B} \models \alpha_0 < y_D$, then $\{\gamma < \lambda | \alpha_0 < y(\gamma)\} \in D$; that is, $\{\gamma < \lambda | \alpha_0 < f_\beta(\gamma)\} \in D$, that is, $\{\alpha < \mu | \alpha_0 < \alpha\} \in D'$, and this implies that D' is uniform over μ, since μ is regular.

(b) $\Rightarrow$ (a). Suppose we have functions $(f_\beta)_{\beta<\kappa}$ as given by (b).

Expand $\langle \lambda, <, \gamma \rangle_{\gamma<\lambda}$ to a model $\mathfrak{A}$ by adding, for each $\beta < \kappa$, a new function symbol representing f_β (by abuse of notation, in what follows we shall write f_β both for the function itself and for the symbol that represents it).

Suppose that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B}$ has an element x such that $\mathfrak{B} \models \gamma < x$ for every $\gamma < \lambda$.

For every formula $\phi(z)$ with just one variable z in the language of $\mathfrak{A}$ let $E_\phi = \{\gamma < \lambda | \mathfrak{A} \models \phi(\gamma)\}$. Let $F = \{E_\phi | \mathfrak{B} \models \phi(x)\}$. Since the intersection of any two members of F is still in F, and $\emptyset \notin F$, then F can be extended to an ultrafilter D on λ.

For every $\gamma_0 < \lambda$, consider the formula $\phi(z) \equiv \gamma_0 < z$. We get $E_\phi = \{\gamma < \lambda | \mathfrak{A} \models \gamma_0 < \gamma\} = (\gamma_0, \lambda)$. On the other side, since $\mathfrak{B} \models \gamma_0 < x$, then by the definition of F we have $E_\phi = (\gamma_0, \lambda) \in F \subseteq D$. Thus, D is uniform over λ.

By (b), $f_\beta(D)$ is uniform over μ, for some $\beta < \kappa$. This means that $(\alpha_0, \mu) \in f_\beta(D)$, for every $\alpha_0 < \mu$. That is, $\{\gamma < \lambda | \alpha_0 < f_\beta(\gamma)\} \in D$ for every $\alpha_0 < \mu$.

For every $\alpha_0 < \mu$, consider the formula $\psi(z) \equiv \alpha_0 < f_\beta(z)$. By the previous paragraph, $E_\psi \in D$. Notice that $E_{\neg\psi}$ is the complement of E_ψ in λ. Since D is proper, and $E_\psi \in D$, then $E_{\neg\psi} \notin D$. Since D extends

F, and either $E_\psi \in F$ or $E_{\neg\psi} \in F$, we necessarily have $E_\psi \in F$, that is, $\mathfrak{B} \models \psi(x)$, that is, $\mathfrak{B} \models \alpha_0 < f_\beta(x)$.

Since $\alpha_0 < \mu$ has been chosen arbitrarily, we have that $\mathfrak{B} \models \alpha_0 < f_\beta(x)$ for every $\alpha_0 < \mu$. Moreover, since $f_\beta : \lambda \to \mu$, and $\mathfrak{B} \equiv \mathfrak{A}$, then $\mathfrak{B} \models f_\beta(x) < \mu$.

Thus, we have proved that $\mathfrak{B}$ has an element $y = f_\beta(x)$ such that $\mathfrak{B} \models \alpha < y < \mu$ for every $\alpha < \mu$.

(b) $\Leftrightarrow$ (b′) follows from Lemma 3 below.

(b′) $\Rightarrow$ (c). Suppose that we have functions $(f_\beta)_{\beta<\kappa}$ as given by (b′). For $\alpha < \mu$ and $\beta < \kappa$, define $B_{\alpha,\beta} = f_\beta^{-1}([0, \alpha))$.

The family $(B_{\alpha,\beta})_{\alpha<\mu,\beta<\kappa}$ trivially satisfies Conditions (i) and (ii). Moreover, Condition (iii) is clearly equivalent to the condition imposed on the f_β's in (b′).

(c) $\Rightarrow$ (b′). Suppose we are given the family $(B_{\alpha,\beta})_{\alpha<\mu,\beta<\kappa}$ from (c). For $\beta < \kappa$ and $\gamma < \lambda$, define $f_\beta(\gamma)$ to be the smallest ordinal $\alpha < \mu$ such that $\gamma \in B_{\alpha,\beta}$ (such an α exists because of (i)).

Because of Condition (ii), we have that $B_{\alpha,\beta} = f_\beta^{-1}([0, \alpha])$, for $\alpha < \mu$ and $\beta < \kappa$. Thus Condition (iii) implies that for every function $g : \kappa \to \mu$ there exists some finite set $F \subseteq \kappa$ such that $\left|\bigcap_{\beta\in F} f_\beta^{-1}([0, g(\beta)])\right| < \lambda$. A fortiori, $\left|\bigcap_{\beta\in F} f_\beta^{-1}([0, g(\beta)))\right| < \lambda$, thus (b′) holds.

The equivalence of Conditions (c)-(e) has been proved in Part I [L7, Theorem 2]. $\square$

Lemma 3. *Suppose that $\lambda \geq \mu$ are infinite regular cardinals, and κ is a cardinal. Suppose that $(f_\beta)_{\beta<\kappa}$ is a given set of functions from λ to μ. Then the following are equivalent.*

(a) Whenever D is an ultrafilter uniform over λ then there exists some $\beta < \kappa$ such that $f_\beta(D)$ is uniform over μ.

(b) For every function $g : \kappa \to \mu$ there exists some finite set $F \subseteq \kappa$ such that $\left|\bigcap_{\beta\in F} f_\beta^{-1}([0, g(\beta)))\right| < \lambda$.

Proof. We show that the negation of (a) is equivalent to the negation of (b).

Indeed, (a) is false if and only if there exists an ultrafilter D uniform over λ such that for every $\beta < \kappa$ $f_\beta(D)$ is not uniform over μ. This means that for every $\beta < \kappa$ there exists some $g(\beta) < \mu$ such that $[g(\beta), \mu) \notin f_\beta(D)$, that is, $f_\beta^{-1}([g(\beta), \mu)) \notin D$, that is, $f_\beta^{-1}([0, g(\beta))) \in D$.

Thus, there exists some D which makes (a) false if and only if there exists some function $g : \kappa \to \mu$ such that the set $\{f_\beta^{-1}([0, g(\beta))) | \beta < \kappa\} \cup \{[\gamma, \lambda) | \gamma < \lambda\}$ has the finite intersection property. Equivalently,

there exists some function $g : \kappa \to \mu$ such that for every $F \subseteq \kappa$ the cardinality of $\bigcap_{\beta \in F} f_\beta^{-1}([0, g(\beta)))$ is equal to λ (since λ is regular).

This is exactly the negation of (b). $\square$

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